Second Part: Residuated Lattices

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Notational preliminaries

A subset F of a poset $\mathbf{P}$ is an *order-filter* of $\mathbf{P}$ if whenever $y \in P$, $x \in F$, and $x \leq y$, then $y \in F$.
For $A \subseteq P$, $\uparrow A = \{x \in P \mid a \leq x, \text{ for some } a \in A\}$ is the *order-filter generated by* A. If $A = \{a\}$, $\uparrow a = \uparrow \{a\}$ is the *principal* order-filter generated by a. *Order-ideals* are defined dually.
$\bot_{\mathbf{P}}$: the *least* element of a poset $\mathbf{P}$ (if it exists). $\top_{\mathbf{P}}$: the *greatest* element.
If $X \subseteq P$, $\bigvee_{\mathbf{P}} X$ and $\bigwedge_{\mathbf{P}} X$ are, respectively, the *join* (or least upper bound) and *meet* (or greatest lower bound) of X in $\mathbf{P}$ whenever they exist.
A map $\varphi : P \rightarrow Q$ between posets $\mathbf{P}$ and $\mathbf{Q}$ s.t. for all $x, y \in P$, if $x \leq y$ then $\varphi(x) \leq \varphi(y)$ is *isotone* or *order-preserving*. *Anti-isotone* or *order-reversing* maps are defined dually. The poset subscripts are omitted whenever there is no danger of confusion.

Residuated maps

Let **P** and **Q** be posets. A map $\varphi : P \longrightarrow Q$ is called *residuated* provided there exists a map $\varphi_* : Q \longrightarrow P$ such that $\varphi(x) \leq y \iff x \leq \varphi_*(y)$, for all $x \in P$ and $y \in Q$. We refer to φ_* as the *residual* of φ.
A simple but useful result:

Lemma

1. *If $\varphi : Q \longrightarrow P$ is residuated with residual φ_*, then φ preserves all existing joins in* **P** *and φ_* preserves any existing meets in* **Q**.
2. *Conversely, if* **P** *is a complete lattice and $\varphi : \mathbf{P} \longrightarrow \mathbf{Q}$ preserves all joins, then it is residuated.*

Example: Galois theory

Consider a field $\mathbf{F}$ together with a field extension $\mathbf{L}/\mathbf{F}$. Let:

- $S(\mathbf{L},\mathbf{F})$ be the set of all subfields of $\mathbf{L}$ that contain $\mathbf{F}$;
- for $\mathbf{M} \in S(\mathbf{L},\mathbf{F})$, $\mathbf{Gal}_{\mathbf{M}}(\mathbf{L})$ be the group of all field automorphisms φ of $\mathbf{L}$ such that $\varphi|_M = id$;
- $Sg(\mathbf{Gal}_{\mathbf{F}}(\mathbf{L}))$ be the set of all subgroups of such a group.

Then the maps

$$
\begin{aligned}
f(\mathbf{M}) &= \mathbf{Gal}_{\mathbf{M}}(\mathbf{L}) \\
f_*(\mathbf{H}) &= \{a \in L \mid \varphi(a) = a \text{ for all } \varphi \in H\}
\end{aligned}
$$

induce a residuated pair (f, f_*) between $\mathbf{P} = (S(\mathbf{L},\mathbf{F}), \subseteq)$ and the order dual $\mathbf{Q}^\partial$ of $\mathbf{Q} = (Sg(\mathbf{Gal}_{\mathbf{F}}(\mathbf{L})), \subseteq)$.

Residuated operations

A binary operation $\cdot$ on a poset $\mathbf{P} = (P, \leq)$ is *residuated* if there exist binary operations $\backslash$ and $/$ on P such that for all $x, y, z \in P$,

$$x \cdot y \leq z \text{ iff } x \leq z/y \text{ iff } y \leq x\backslash z.$$

Observe:

1. $\cdot$ is residuated if and only if, for all $a \in P$, the maps $x \mapsto ax$ $(x \in P)$ and $x \mapsto xa$ $(x \in P)$ are residuated in the sense of the preceding definition. Their residuals are the maps $y \mapsto a\backslash y$ $(y \in P)$ and $y \mapsto y/a$ $(y \in P)$, respectively. $\backslash$ and $/$ are the *right residual* and the *left residual* of $\cdot$, respectively.
2. $\cdot$ is residuated if and only if it is order-preserving in each argument and, for all $x, y \in P$, the sets $\{z \mid x \cdot z \leq y\}$ and $\{z \mid z \cdot x \leq y\}$ both contain greatest elements, $x\backslash y$ and y/x, respectively.
3. $\cdot$ and $\leq$ uniquely determine $\backslash$ and $/$.

Terminology and notational conventions

We write xy for $x \cdot y$, x^2 for xx and adopt the convention that, in the absence of parentheses, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally by $\wedge$ and $\vee$.

The residuals may be viewed as generalized division operations, with x/y being read as "x over y" and $y\backslash x$ as "y under x". In either case, x is considered the *numerator* and y is the *denominator*.

They can also be viewed as generalized implication operators, with x/y being read as "x if y" and $y\backslash x$ as "if y then x". In either case, x is considered the *consequent* and y is the *antecedent*.

Any statement about residuated structures has a "mirror image" obtained by reading terms backwards (i.e., replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y\backslash x$).

Preservation of meets and joins

Lemma

Let $\cdot$ be a residuated operation on a poset $\mathbf{P}$ *with residuals* $\backslash$ *and* $/$.

1. $\cdot$ *preserves all existing joins in each argument; i.e., if* $\bigvee X$ *and* $\bigvee Y$ *exist for* $X, Y \subseteq A$*, then* $\bigvee_{x \in X, y \in Y}(xy)$ *exists and*

$$\left(\bigvee X\right)\left(\bigvee Y\right) = \bigvee_{x \in X, y \in Y}(xy).$$

2. $\backslash$ *and* $/$ *preserve all existing meets in the numerator, and convert existing joins to meets in the denominator, i.e. if* $\bigvee X$ *and* $\bigwedge Y$ *exist for* $X, Y \subseteq A$*, then for any* $z \in A$*,* $\bigwedge_{x \in X}(x \backslash z)$ *and* $\bigwedge_{y \in Y}(z \backslash y)$ *exist and*

$$\left(\bigvee X\right) \Big\backslash z = \bigwedge_{x \in X}(x \backslash z) \text{ and } z \Big\backslash \left(\bigwedge Y\right) = \bigwedge_{y \in Y}(z \backslash y).$$

Residuated lattices

A *residuated lattice* is an algebra

$$\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$$

such that:

1. $(L, \wedge, \vee)$ is a lattice;
2. $(L, \cdot, 1)$ is a monoid;
3. $\cdot$ is residuated, in the underlying partial order, with residuals $\backslash, /$.

An *FL-algebra* $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ is an algebra s.t. $(L, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated lattice and 0 is a distinguished element (nullary operation) of L.
We use the symbols $\mathcal{RL}$ and $\mathcal{FL}$ to denote the class of all residuated lattices and FL-algebras respectively.

RL and FL are varieties

Proposition

$\mathcal{RL}$ and $\mathcal{FL}$ are finitely based varieties in their respective languages: the residuation condition (3) can be replaced by the following equations (and their mirror images):

(i) $y \leq x\backslash(xy \vee z)$

(ii) $x(y \vee z) \approx xy \vee xz$

(iii) $y(y\backslash x) \leq x$

Commutative residuated lattices

Two varieties of particular interest are the variety $\mathcal{CRL}$ of commutative residuated lattices and the variety $\mathcal{CFL}$ of commutative FL-algebras. These varieties satisfy the equation $xy \approx yx$, and hence the equation $x\backslash y \approx y/x$. Here, $\rightarrow$ can denote both $\backslash$ and $/$.
We always think of these varieties as subvarieties of $\mathcal{RL}$ and $\mathcal{FL}$, respectively, but we slightly abuse notation by listing only one occurrence of the operation $\rightarrow$ in describing their members.

Example: Boolean algebras and Heyting algebras

Example

The variety of *Boolean algebras* is term-equivalent to the subvariety $\mathcal{BA}$ of $\mathcal{CFL}$ satisfying the additional equations $xy \approx x \wedge y$, $(x \rightarrow y) \rightarrow y \approx x \vee y$, and $x \wedge 0 \approx 0$. More specifically, every Boolean algebra $\mathbf{B} = (B, \wedge, \vee, \neg, 1, 0)$ satisfies the equations above with respect to $\wedge, \vee, 0$, and Boolean implication $x \rightarrow y = \neg x \vee y$. Conversely, if a (commutative) residuated lattice $\mathbf{L}$ satisfies these equations and we define $\neg x = x \rightarrow 0$, then $(L, \wedge, \vee, \neg, 1, 0)$ is a Boolean algebra.

Likewise, the variety of *Heyting algebras* is term-equivalent to the subvariety $\mathcal{HA}$ of $\mathcal{CFL}$ satisfying the additional equations $xy \approx x \wedge y$ and $x \wedge 0 \approx 0$.

Example: Ideals of rings

Example

Let $\mathbf{R}$ be a ring with unit and let $I(\mathbf{R})$ denote the set of two-sided ideals of $\mathbf{R}$. Then $\mathbf{I}(\mathbf{R}) = (I(\mathbf{R}), \cap, \vee, \cdot, \backslash, /, R, \{0\})$ is a (not necessarily commutative) FL-algebra, where, for $I, J \in I(\mathbf{R})$,

$$IJ = \{\sum_{k=1}^{n} a_k b_k : a_k \in I; b_k \in J; n \geq 1\};$$

$$I \backslash J = \{x \in R : Ix \subseteq J\}; \text{ and}$$

$$J/I = \{x \in R : xI \subseteq J\}.$$

Example: Lattice-ordered groups

Example

A *lattice-ordered group* (ℓ*-group*) is an algebra $\mathbf{G} = \left(G, \wedge, \vee, \cdot, ^{-1}, 1 \right)$ such that (i) $(G, \wedge, \vee)$ is a lattice; (ii) $\left(G, \cdot, ^{-1}, 1 \right)$ is a group; and (iii) multiplication is order-preserving in each argument. The variety of ℓ-groups is term-equivalent to the subvariety $\mathcal{LG}$ of $\mathcal{RL}$ defined by the additional equation $(1/x)x \approx 1$.

More specifically, if $\mathbf{G} = \left(G, \wedge, \vee, \cdot, ^{-1}, 1 \right)$ is an ℓ-group and we define $x/y = xy^{-1}$, $y \backslash x = y^{-1}x$, then $\mathbf{G} = (G, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated lattice satisfying the equation $(1/x)x \approx 1$. Conversely, if a residuated lattice $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ satisfies the last equation and we define $x^{-1} = 1/x$, then $\mathbf{L} = \left(L, \wedge, \vee, \cdot, ^{-1}, 1 \right)$ is an ℓ-group. Moreover, this correspondence is bijective.

Example: MV algebras

Example

MV-algebras are the algebraic counterparts of the infinite-valued Łukasiewicz propositional logic. An *MV-algebra* is traditionally defined as an algebra $\mathbf{M} = (M, \oplus, \neg, 0)$ of language $(2, 1, 0)$ that satisfies the following equations:

(MV1)	$x \oplus (y \oplus z) \approx (x \oplus y) \oplus z$
(MV2)	$x \oplus y \approx y \oplus x$
(MV3)	$x \oplus 0 \approx x$
(MV4)	$\neg\neg x \approx x$
(MV5)	$x \oplus \neg 0 \approx \neg 0$
(MV6)	$\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$

The variety of MV-algebras is term-equivalent to the subvariety, $\mathcal{MV}$, of $\mathcal{CFL}$ satisfying the extra equations $x \vee y \approx (x \to y) \to y$ and $x \wedge 0 \approx 0$.

Example: Semilinear residuated lattices

Example

Many varieties of ordered algebras arising in logic – including Boolean algebras, Abelian ℓ-groups, and MV-algebras, but not Heyting algebras and ℓ-groups – are *semilinear*, that is, generated by their totally ordered members. An equational basis for the variety $\mathcal{S}em\mathcal{RL}$ of semilinear residuated lattices relative to $\mathcal{RL}$ consists of the equation

$$\lambda_z(x/(x \vee y)) \vee \rho_w(y/(x \vee y)) \approx 1,$$

where $\rho_w(x) = wx/w \wedge 1$, $\lambda_z(x) = z \backslash xz \wedge 1$.
A simplified equational basis for the variety $\mathcal{CS}em\mathcal{RL}$ of commutative semilinear residuated lattices relative to $\mathcal{CRL}$ consists of the equations

$$[(x \to y) \vee (y \to x)] \wedge 1 \approx 1 \quad \textit{and} \quad 1 \wedge (x \vee y) \approx (1 \wedge x) \vee (1 \wedge y).$$

Historical note

- Morgan Ward and his student R.P. Dilworth, in a series of papers from the late 1930's, introduced under the name of *residuated lattices* some lattice-ordered structures with a multiplication abstracted from ideal multiplication, and with a residuation abstracted from ideal residuation.
- Ward and Dilworth's papers did not have that much immediate impact. The notion of residuated lattice re-emerged later in the different contexts of the semantics for fuzzy logics (P. Hàjek) and substructural logics (H. Ono), and in the setting of studies with a more pronounced algebraic flavor (K. Blount, C. Tsinakis), with the latter two streams eventually converging into a single one.

Historical note

- Neither Hájek's nor Ono's, nor our official definition of residuated lattice (essentially due to Blount and Tsinakis), exactly overlaps with the original definition given by Ward and Dilworth. The Hájek-Ono residuated lattices are *bounded* as lattices and *integral* as partially ordered monoids. Moreover, multiplication is *commutative*. Residuated lattices as defined here need not be bounded or integral; multiplication is not required to be commutative. The original Ward-Dilworth residuated lattices lie somewhere in between: the existence of a top or bottom element is not assumed, but if there is a top, then it must be the neutral element of multiplication, which is supposed to be a commutative operation.
- Dilworth also introduced a noncommutative variant of his notion of residuated lattice, abstracted from the residuated lattice of two-sided ideals of a noncommutative ring, but to the best of our knowledge this generalization was not taken up again before the work by Tsinakis and his collaborators.

RL is an ideal variety

The congruences of residuated lattices are determined by special subsets of their universes (like for groups or rings). In particular, $\mathcal{RL}$, and hence $\mathcal{FL}$, is a 1-*regular variety*: each congruence of an algebra in $\mathcal{RL}$ is determined by its equivalence class of 1. A more economical proof of 1-regularity for $\mathcal{RL}$ can be given by observing that this property is a *Mal'cev property*: one can establish if a variety has the property by checking whether it satisfies certain quasi-equations involving finitely many terms (two, in this special case).
However, a concrete description of these equivalence classes is essential for developing the structure theory of residuated lattices and its applications to substructural logics.

Some terminology

If $\mathbf{L}$ is a residuated lattice, the set $L^{-} = \{a \in L \mid a \leq 1\}$ is called the *negative cone* of $\mathbf{L}$. Note that the negative cone is a submonoid of $(L, \cdot, 1)$. As such, we will denote it by $\mathbf{L}^{-}$.
Let $\mathbf{L} \in \mathcal{RL}$. Recall that for each $a \in L$, $\rho_a(x) = ax/a \wedge 1$ and $\lambda_a(x) = a\backslash xa \wedge 1$. We refer to ρ_a and λ_a respectively as *right and left conjugation* by a. An *iterated conjugation* map is a finite composition of right and left conjugation maps.
A subset $X \subseteq L$ is *convex* if for any $x, y \in X$ and $a \in L$, $x \leq a \leq y$ implies $a \in X$; X is *normal* if it is closed with respect to all iterated conjugations.

Two useful lemmas

Lemma

Let $\mathbf{L}$ *be a residuated lattice and* $\Theta \in Con(\mathbf{L})$. *T.f.a.e.:*

1. $a \, \Theta \, b$
2. $[a/b \wedge 1] \, \Theta \, 1$ *and* $[b/a \wedge 1] \, \Theta \, 1$
3. $[a \backslash b \wedge 1] \, \Theta \, 1$ *and* $[b \backslash a \wedge 1] \, \Theta \, 1$

Lemma

Suppose that $\mathbf{H}$ *is a convex normal subalgebra of* $\mathbf{L}$. *For any* $a, b \in L$,

$$a/b \wedge 1 \in H \;\Leftrightarrow\; b \backslash a \wedge 1 \in H.$$

From congruences to convex normal subalgebras

Lemma

Let Θ be a congruence on a residuated lattice **L**. *Then* $[1]_\Theta = \{a \in A \mid a \,\Theta\, 1\}$ *is a convex normal subalgebra of* **L**.

Proof.

Since 1 is idempotent with respect to all the binary operations of **L**, $[1]_\Theta$ forms a subalgebra of **L**. Convexity is a consequence of the fact that the equivalence classes of any lattice congruence are convex. Finally, let $a \in [1]_\Theta$ and $c \in L$. Then

$$\lambda_c(a) = c \backslash ac \wedge 1 \;\Theta\; c \backslash 1c \wedge 1 = c \backslash c \wedge 1 = 1$$

so that $\lambda_c(a) \in [1]_\Theta$. Similarly, $\rho_c(a) \in [1]_\Theta$. □

From convex normal subalgebras to congruences (1)

Lemma

Let **H** *be a convex normal subalgebra of a residuated lattice* **L**. *Then the following is a congruence on* **L**:

$$\begin{aligned}\Theta_{\mathbf{H}} &= \{(a,b) \mid \exists h \in H, ha \leq b \text{ and } hb \leq a\} \\ &= \{(a,b) \mid a/b \wedge 1 \in H \text{ and } b/a \wedge 1 \in H\} \\ &= \{(a,b) \mid a\backslash b \wedge 1 \in H \text{ and } b\backslash a \wedge 1 \in H\}\end{aligned}$$

Proof.

The second and third set are equal by the useful Lemma. If (a,b) is a member of the second set, letting $h = a/b \wedge b/a \wedge 1$, we have $h \in H$, $ha \leq (b/a)a \leq b$ and $hb \leq (a/b)b \leq a$, so (a,b) is a member of the first set. Conversely, if (a,b) is a member of the first set, for some $h \in H$ we have $ha \leq b$ or $h \leq b/a$, whence $h \wedge 1 \leq b/a \wedge 1 \leq 1$. By convexity, we get $b/a \wedge 1 \in H$. Similarly, $a/b \wedge 1 \in H$. □

From convex normal subalgebras to congruences (2)

Proof.

(continued) It is a simple matter to verify that $\Theta_{\mathbf{H}}$ is an equivalence. To prove that it is a congruence, we must establish its compatibility with respect to multiplication, meet, join, right division, and left division. We just verify compatibility for multiplication. Suppose that $a \; \Theta_{\mathbf{H}} \; b$ and $c \in L$. Then

$$a/b \wedge 1 \leq ac/bc \wedge 1 \leq 1$$

so $ac/bc \wedge 1 \in H$. Similarly, $bc/ac \wedge 1 \in H$ so $(ac) \; \Theta_{\mathbf{H}} \; (bc)$. Next, using the normality of $\mathbf{H}$,

$$\rho_c(a/b \wedge 1) = (c[a/b \wedge 1]/c) \wedge 1 \in H.$$

But
$\rho_c(a/b \wedge 1) \leq [c(a/b)/c] \wedge 1 \leq (ca/b)/c \wedge 1 = ca/cb \wedge 1 \leq 1 \in H$ so $ca/cb \wedge 1 \in H$. Similarly, $cb/ca \wedge 1 \in H$ so $(ca) \; \Theta_{\mathbf{H}} \; (cb)$. □

The bijective correspondence (1)

Theorem

The lattice $CN(\mathbf{L})$ of convex normal subalgebras of a residuated lattice $\mathbf{L}$ is isomorphic to its congruence lattice $Con(\mathbf{L})$. The isomorphism is given by the mutually inverse maps $\mathbf{H} \mapsto \Theta_{\mathbf{H}}$ and $\Theta \mapsto [1]_{\Theta}$.

Proof.

We have shown both that $\Theta_{\mathbf{H}}$ is a congruence and that $[1]_{\Theta}$ is a member of $CN(\mathbf{L})$, and it is clear that the maps $\mathbf{H} \mapsto \Theta_{\mathbf{H}}$ and $\Theta \mapsto [1]_{\Theta}$ are isotone. It remains only to show that these two maps are mutually inverse, since it will then follow that they are lattice homomorphisms. $\square$

The bijective correspondence (2)

Proof.

(continued) Given $\Theta \in Con(\mathbf{L})$, set $H = [1]_\Theta$; we must show that $\Theta = \Theta_\mathbf{H}$. But this is easy; using the above Lemma,

$$a \; \Theta \; b \; \Leftrightarrow (a/b \wedge 1) \; \Theta \; 1 \text{ and } (b/a \wedge 1) \; \Theta \; 1 \Leftrightarrow$$

$$a/b \wedge 1 \in H \text{ and } b/a \wedge 1 \in H \Leftrightarrow a \; \Theta_\mathbf{H} \; b.$$

Conversely, for any $\mathbf{H} \in CN(\mathbf{L})$ we must show that $H = [1]_{\Theta_\mathbf{H}}$. But

$$h \in H \Rightarrow h/1 \wedge 1 \in H \text{ and } 1/h \wedge 1 \in H$$

so $h \in [1]_{\Theta_\mathbf{H}}$. If $a \in [1]_{\Theta_\mathbf{H}}$, then $(a, 1) \in \Theta_\mathbf{H}$ and we use the first description of $\Theta_\mathbf{H}$ above to conclude that there exists some $h \in H$ such that $ha \leq 1$ and $h = h1 \leq a$. Now it follows from the convexity of $\mathbf{H}$ that $h \leq a \leq h \backslash 1$ implies $a \in H$. □

The commutative case

We remark that in the event that **L** is commutative, then every convex subalgebra of **L** is normal. Thus the preceding theorem implies the following result:

Corollary

The lattice $\mathcal{C}(\mathbf{L})$ *of convex subalgebras of a commutative residuated lattice* **L** *is isomorphic to its congruence lattice* $Con(\mathbf{L})$*. The isomorphism is given by the mutually inverse maps* $\mathbf{H} \mapsto \Theta_{\mathbf{H}}$ *and* $\Theta \mapsto [1]_{\Theta}$*.*